

Stability estimate for the relativistic Schrödinger equation with time-dependent vector potentials

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Abstract

We consider the relativistic Schrödinger equation with a time dependent vector and scalar potential on a bounded cylindrical domain. Using a Geometric Optics Ansatz we establish a logarithmic stability estimate for the recovery of the vector potentials.

1 Introduction

Let Ω be a bounded domain in \mathbb{R}^n , $n \geq 2$, consider the hyperbolic equation with time dependent coefficients

$$\left(-i\partial_t + A_0(t, x)\right)^2 u - \sum_{j=1}^n \left(-i\partial_{x_j} + A_j(t, x)\right)^2 u + V(t, x)u = 0 \quad \text{in } \mathbb{R} \times \Omega, \quad (1)$$

where $A_j(t, x)$, $0 \leq j \leq n$, and $V(t, x)$ are compactly supported smooth functions.

The vector field $\mathcal{A}(t, x) = (A_0(t, x), \dots, A_n(t, x))$ is called the *vector potential*, the function $V(t, x)$ is called the *scalar potential* and equation (1) is often referred to as the *relativistic Schrödinger equation* or, in the case where the vector potential is zero and the scalar potential is proportional to the mass of a free particle, it is referred to as the *Klein-Gordon equation* (see [26]).

We impose the initial and boundary conditions

$$u(t, x) = \partial_t u(t, x) = 0 \quad \text{for } t \ll 0 \quad (2)$$

$$u(t, x) = f(t, x) \quad \text{on } \mathbb{R} \times \partial\Omega, \quad (3)$$

where f is a compactly supported smooth function on $\mathbb{R} \times \partial\Omega$. Solutions to (1) satisfying (2) and (3) exist and are unique (Theorem 8.1 in [17]) and we can define the *Dirichlet to Neumann* operator by

$$\Lambda(f) := (\partial_\nu + iA(t, x) \cdot \nu) u(t, x) \Big|_{\mathbb{R} \times \partial\Omega} \quad (4)$$

where u is the solution of (1)-(3), ν is the exterior unit normal to $\partial\Omega$ and we have set $A(t, x) = (A_1(t, x), \dots, A_n(t, x))$. The *Inverse Boundary Value Problem* is the recovery of $\mathcal{A}(t, x)$ and $V(t, x)$ knowing $\Lambda(f)$ for all $f \in C_0^\infty(\mathbb{R} \times \partial\Omega)$.

Definition 1.1. The pair $(\mathcal{A}(t, x), V(t, x))$ and $(\mathcal{A}'(t, x), V'(t, x))$ are said to be *gauge equivalent* if there exists $g(t, x) \in C^\infty(\mathbb{R} \times \overline{\Omega})$ such that $g(t, x) \neq 0$ on $\mathbb{R} \times \overline{\Omega}$, $g = 1$ on $\mathbb{R} \times \partial\Omega$ and

$$\begin{aligned} \mathcal{A}'(t, x) &= \mathcal{A}(t, x) - \frac{i}{g(t, x)} \nabla_{t,x} g(t, x) \\ V'(t, x) &= V(t, x), \end{aligned}$$

where $\nabla_{t,x} := (\partial_t, \partial_x) = (\partial_t, \partial_{x_1}, \dots, \partial_{x_n})$ is the $(n+1)$ -dimensional gradient. The mapping $(\mathcal{A}, V) \rightarrow (\mathcal{A}', V')$ is called a *gauge transform*. The Dirichlet to Neumann maps Λ and Λ' are said to be *gauge equivalent* if for all $f(t, x) \in C_0^\infty(\mathbb{R} \times \partial\Omega)$,

$$\Lambda'(g(t, x)f(t, x)) = g(t, x)\Lambda(f(t, x)). \quad (5)$$

Remark: When Ω is simply connected, the gauge g has the particular form $g(t, x) = e^{i\varphi(t, x)}$ where $\varphi(t, x) \in C^\infty(\mathbb{R} \times \Omega)$. Then $-\frac{i}{g(t, x)} \nabla_{(t,x)} g(t, x) = \nabla_{(t,x)} \varphi(t, x)$ and two vector potentials are gauge equivalent if their difference is the gradient of a smooth function.

Inverse problems is a topic in mathematics that has been growing in interest in part, due to its wide range of applications, from medicine to acoustics to electromagnetism (see for instance [17] for some of the latest tools and

techniques employed in the solutions of these problems). In the case of the hyperbolic inverse boundary value problem (1)-(4) with time independent coefficients, a powerful tool called the *boundary control method*, or BC-method for short, was discovered by Belishev (see [3]). It was later developed by Belishev, Kurylev, Lassas, and others ([19],[20]). A new approach to this problem based on the BC-method was developed by Eskin in ([9],[10]). On a similar note, Stefanov and Uhlmann established stability results for the wave equation in anisotropic media (see [28], [29] and [31] for a survey of these results).

In the case where the scalar potential is time-dependent and the vector potential is identically equal to zero ($\mathcal{A} \equiv 0$ in (1)), Stefanov [27] and Ramm-Sjöstrand [22], have shown that the Dirichlet to Neumann map completely determines the scalar potentials. In [11], Eskin considered the case with time-dependent potentials that are analytic in time (this case is more general in terms of the complexity of the PDE but less general with its assumption of analyticity). The analyticity of the time variable is related to the use of a unique continuation theorem established by Tataru in [30]. More recently, the results of [22, 27] were generalized by the author in [23, 24] for the case of vector potentials, where it was shown that the Dirichlet-to-Neumann operator determines the vector and scalar potentials up to a gauge transform.

Regarding elliptic problems, the questions of stability and reconstruction have been studied for several IBVP (see [1, 4, 7, 8, 17, 31] and the references therein). For the parabolic case, there are a few results concerning the determination of time-dependent coefficients in an IBVP. The case of a source term of the form $f(t)\chi_D$, where χ_D is the characteristic function of a known subdomain was considered by Perez-Esteva and Canon in [5] in a half line in one dimension. Later in [6] they considered a similar problem in 3 dimensions. For more references on recent developments on uniqueness and stability estimates on elliptic and parabolic PDE's the reader is referred to the books by Isakov [17] and Choulli [4].

Regarding the stability in the hyperbolic case, the first results were obtained by Isakov in [14]. Isakov and Sun [15] obtained estimates for two coefficients of a hyperbolic partial differential equation from all measurements on a part of the lateral boundary. In [28] Stefanov and Uhlmann studied the hyperbolic Dirichlet to Neumann map associated to the wave equation in anisotropic media; and in [29], they consider the more general case of determining a Riemannian metric on a Riemannian manifold with boundary from the boundary measurements. More recently in [21], Montalto recovers

the metric, a covector field and a potential from the hyperbolic Dirichlet to Neumann map.

However, stability in the case of time-dependent vector has not been considered before. In this paper, which is based on [24, 25], we take advantage of a result by Begmatov [2], where he proves a stability estimate for a time-dependent scalar function when information about its X-ray transforms is known on a cone. In our work we establish stability estimates for vector and scalar potentials when they are compactly supported in space and time. This work is structured as follows. In section 2 we review the construction of the Geometric Optics Ansatz (GO for short) as well as the Green's formula developed in [23]. This construction is later used to obtain estimates for the X-ray transform along 'light rays' of particular combinations of the components of the vector potentials. In section 3 a logarithmic stability estimate for vector potentials is established, and finally in section 4 we prove an estimate for the case when both vector and scalar potentials are present.

2 Geometric optics and Green's formula

The following Geometric Optics construction is the same in [23], however, it is included here because some of the details will be needed in the estimates in section 3.

For the hyperbolic problem (1)-(3) Geometric Optics Ansatz supported near light rays take the form

$$u(t, x) = e^{ik(t-\omega \cdot x)} \sum_{p=0}^N \frac{v_p(t, x)}{(2ik)^p} + v^{(N+1)}(t, x), \quad \omega \in S^{n-1}, k \in \mathbb{R}. \quad (6)$$

For u as above, equation (1) yields (see [23] for the details)

$$0 = (L + 2ik\mathcal{L})v, \quad (7)$$

where

$$v(t, x) = \sum_{p=0}^N \frac{v_p(t, x)}{(2ik)^p} + e^{-ik(t-\omega \cdot x)} v^{(N+1)}(t, x) \quad (8)$$

$$L = (-i\partial_t + A_0(t, x))^2 - \sum_{j=1}^n (-i\partial_{x_j} + A_j(t, x))^2 + V(t, x) \quad (9)$$

$$\mathcal{L} = -(\partial_t + iA_0(t, x)) - \sum_{j=1}^n \omega_j (\partial_{x_j} + iA_j(t, x)), \quad (10)$$

plugging in v into (7) then gives

$$\begin{aligned} (2ik)\mathcal{L}v_0 + (\mathcal{L}v_1 + Lv_0) + \frac{1}{(2ik)}(\mathcal{L}v_2 + Lv_1) + \cdots + \\ \frac{1}{(2ik)^{N-1}}(\mathcal{L}v_N + Lv_{N-1}) + \frac{1}{(2ik)^N}Lv_N + e^{-ik(t-\omega \cdot x)}Lv^{(N+1)} = 0. \end{aligned} \quad (11)$$

To ensure that the previous equation is satisfied, we can use a two-step process. In the first step we solve the $N+1$ transport equations

$$\mathcal{L}v_0 = 0, \quad \mathcal{L}v_j = -Lv_{j-1}, \quad 1 \leq j \leq N \quad (12)$$

with initial conditions supported in a small neighborhood of a point $(t, x) \in \mathbb{R} \times \partial\Omega$, and in the second step we solve the second order equation

$$Lv^{(N+1)} = -\frac{e^{ik(t-\omega \cdot x)}}{(2ik)^N}Lv_N \quad (13)$$

with initial and boundary conditions

$$\begin{aligned} v^{(N+1)}(t, x) &= 0 & \text{for } t = T_1 \\ \partial_t v^{(N+1)}(t, x) &= 0 & \text{for } t = T_1 \\ v^{(N+1)}(t, x) &= 0 & \text{for } t \geq T_1, \quad x \in \partial\Omega. \end{aligned}$$

This differential equation admits a unique solution; moreover if we denote by h the right hand side of (13), then for $T_1 < t < T$ and $k > 1$ (see [17], pp. 185)

$$\begin{aligned} \|\partial_t v^{(N+1)}(t, \cdot)\|_{L^2(\Omega)} + \|v^{(N+1)}(t, \cdot)\|_{H^1(\Omega)} &\leq C\|h\|_{L^2((T_1, T) \times \Omega)} \\ &\leq \frac{C}{k^N}. \end{aligned} \quad (14)$$

If v_0 is a solution of the transport equation $\mathcal{L}v_0(t, x) = 0$, it has the form

$$v_0(t, x) = \chi_1(t', x') \exp \left[-i \int_{-\infty}^{(t+\omega \cdot x)/2} \sum_{j=0}^n \omega_j A_j(t' + s, x' + s\omega) ds \right] \quad (15)$$

where $(t', x') = (t, x) - \frac{1}{2}(t + \omega \cdot x)(1, \omega)$ is the projection of (t, x) into $\Pi_{(1, \omega)}$, the n -dimensional linear subspace perpendicular to $(1, \omega)$ (see figure 1 in [23]), and χ_1 is any real valued function that is constant along the direction given by $(1, \omega)$, and whose support is contained in a neighborhood of the light ray $\gamma = \{(t', x') + s(1, \omega) \mid s \in \mathbb{R}\}$.

It then follows that $u = e^{ik(t - \omega \cdot x)}(v_0 + \mathcal{O}(k^{-1}))$ solves (1) and satisfies the set of initial conditions (2). Summarizing, a GO solution of (1)-(2) of the form

$$u(t, x) = \exp [ik(t - \omega \cdot x) - iR_1(t, x; \omega)] (\chi_1(t', x') + \mathcal{O}(k^{-1})) \quad (16)$$

can be constructed, where

$$R_1(t, x; \omega) = \int_{-\infty}^{(t+\omega \cdot x)/2} \sum_{j=0}^n \omega_j A_j(t' + s, x' + s\omega) ds. \quad (17)$$

Similarly, a GO solution for the backwards hyperbolic problem can be obtained in the same fashion, with another real valued function χ_2 constant along a given light ray.

To obtain a Green's formula for this problem, we let T_1 and T_2 be two real numbers with $T_1 \ll 0 \ll T_2$, and consider the forward and backward hyperbolic equations

$$\begin{array}{llll} L_1 u = 0 & \text{in } [T_1, T_2] \times \Omega & L_2^* v = 0 & \text{in } [T_1, T_2] \times \Omega \\ u = \partial_t u = 0 & \text{for } t = T_1 & v = \partial_t v = 0 & \text{for } t = T_2 \\ u = f & \text{on } [T_1, T_2] \times \partial\Omega & v = g & \text{on } [T_1, T_2] \times \partial\Omega, \end{array}$$

where

$$\begin{aligned} L_1 &= (-i\partial_t + A_0^{(1)}(t, x))^2 - \sum_{j=1}^n (-i\partial_{x_j} + A_j^{(1)}(t, x))^2 + V^{(1)}(t, x) \\ L_2^* &= (-i\partial_t + \overline{A_0^{(2)}(t, x)})^2 - \sum_{j=1}^n (-i\partial_{x_j} + \overline{A_j^{(2)}(t, x)})^2 + \overline{V^{(2)}(t, x)}. \end{aligned}$$

If we denote by $\langle \cdot, \cdot \rangle_{[T_1, T_2] \times \Omega}$ and $\langle \cdot, \cdot \rangle_{[T_1, T_2] \times \partial\Omega}$ the L^2 inner products in $[T_1, T_2] \times \Omega$, $[T_1, T_2] \times \partial\Omega$; integration by parts applied to

$$\langle L_1 u, v \rangle_{[T_1, T_2] \times \Omega} - \langle u, L_2^* v \rangle_{[T_1, T_2] \times \Omega} = 0$$

yields the Green's Formula (see [23] for complete details)

$$\begin{aligned} & \langle \Lambda_1(f), g \rangle_{[T_1, T_2] \times \partial\Omega} - \langle \Lambda_2(f), g \rangle_{[T_1, T_2] \times \partial\Omega} = \\ & \sum_{j=0}^n r_j \left(\langle A_j u, (-i\partial_{x_j} v) \rangle_{[T_1, T_2] \times \Omega} + \langle A_j (-i\partial_{x_j} u), v \rangle_{[T_1, T_2] \times \Omega} \right) \\ & + \sum_{j=0}^n r_j \langle ((A_j^{(2)})^2 - (A_j^{(1)})^2) u, v \rangle_{[T_1, T_2] \times \Omega} - \langle V u, v \rangle_{[T_1, T_2] \times \Omega}, \quad (18) \end{aligned}$$

where $x_0 = t$, $A_j = A_j^{(2)} - A_j^{(1)}$ for $0 \leq j \leq n$, $V = V^{(2)} - V^{(1)}$, $r_0 = -1$, and $r_j = 1$ for $1 \leq j \leq n$.

3 Stability of the vector potentials

The proof in this section closely follows [24]. We assume that the components of the vector potentials $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ as well as the scalar potentials $V^{(1)}$ and $V^{(2)}$ are real valued, smooth and compactly supported in both t and x . We write

$$\mathcal{A} = \mathcal{A}^{(1)} - \mathcal{A}^{(2)} \quad \text{where} \quad \mathcal{A}^{(k)} = (A_0^{(k)}, \dots, A_n^{(k)}), \quad k = 1, 2,$$

and as before we denote by $\Pi_{(1, \omega)}$ the n -dimensional linear subspace perpendicular to $(1, \omega)$. In symbols

$$\Pi_{(1, \omega)} = \{(t, x) : t + \omega \cdot x = 0\}.$$

The GO anzats and the Green's formula developed in the previous section allow for the estimation of the X-ray transform over light rays of particular combinations of the components of the vector potentials. The precise statement is as follows:

Lemma 3.1. *If Λ_k , $k = 1, 2$ represents the Dirichlet to Neumann operator for the hyperbolic equations*

$$\left((-i\partial_t + A_0^{(k)}(t, x))^2 - \sum_{j=1}^n (-i\partial_{x_j} + A_j^{(k)}(t, x))^2 + V^{(k)}(t, x) \right) u = 0, \quad (19)$$

then for all $(t, x) \in \mathbb{R}_{t,x}^{n+1}, \omega \in S^{n-1}$, the vectorial ray transform of $\mathcal{A} = (A_0^{(2)} - A_0^{(1)}, \dots, A_n^{(2)} - A_n^{(1)})$ along the light rays

$$\gamma_{(t,x;\omega)} = \{(t, x) + s(1, \omega) : s \in \mathbb{R}\},$$

satisfies

$$\left| \exp \left[i \int_{-\infty}^{\infty} (A_0 + \sum_{j=1}^n \omega_j A_j)(t + s, x + s\omega) ds \right] - 1 \right| \leq C ||| \Lambda_1 - \Lambda_2 |||, \quad (20)$$

where $||| \cdot |||$ represents the operator norm between $H^1([T_1, T_2] \times \partial\Omega)$ and $L^2([T_1, T_2] \times \partial\Omega)$.

Remark: We point that this result is independent of the presence of scalar potentials.

Proof. Owing to (16) and (17), GO Ansatz for the forward and backward hyperbolic equations are given by

$$u(t, x) = \exp \left[ik(t - \omega \cdot x) - iR_1(t, x; \omega) \right] \left(\chi_1 + \mathcal{O}(k^{-1}) \right), \quad (21)$$

$$\overline{v(t, x)} = \exp \left[-ik(t - \omega \cdot x) + i\overline{R_2(t, x; \omega)} \right] \left(\chi_2 + \mathcal{O}(k^{-1}) \right), \quad (22)$$

where

$$R_1(t, x; \omega) = \int_{-\infty}^{(t+\omega \cdot x)/2} \sum_{j=0}^n \omega_j A_j^{(1)}(t' + s, x' + s\omega) ds, \quad (23)$$

$$\overline{R_2(t, x; \omega)} = \int_{-\infty}^{(t+\omega \cdot x)/2} \sum_{j=0}^n \omega_j A_j^{(2)}(t' + s, x' + s\omega) ds, \quad (24)$$

where χ_1, χ_2 are constant along, and supported on a small neighborhood of the light ray $\gamma_{(t,x;\omega)}$, and where (t', x') is the projection of (t, x) into $\Pi_{(1,\omega)}$.

For $0 \leq j \leq n$, differentiation of (6) with respect to x_j combined with estimate (14) lead to

$$\partial_{x_j} u = k \exp \left[ik(t - \omega \cdot x) - iR_1(t, x; \omega) \right] \left(-ir_j \omega_j \chi_1 + \mathcal{O}(k^{-1}) \right), \quad (25)$$

where $x_0 = t, \omega_0 = 1, r_0 = -1$ and $r_j = 1$ when $j \neq 0$. Then by (22)

$$(-i\partial_{x_j} u(t, x)) \overline{v(t, x)} = -ke^{i(\overline{R_2(t,x;\omega)} - R_1(t,x;\omega))} (r_j \omega_j \chi_1 \chi_2 + \mathcal{O}(k^{-1})). \quad (26)$$

Similarly, (21) yields

$$u(t, x) \overline{(-i\partial_{x_j} v(t, x))} = -k e^{i(\overline{R_2(t, x; \omega)} - R_1(t, x; \omega))} (r_j \omega_j \chi_1 \chi_2 + \mathcal{O}(k^{-1})). \quad (27)$$

Denoting by \mathcal{I}_R the right hand side of (18), we obtain via the previous two formulas

$$\begin{aligned} \mathcal{I}_R = Ck \int_{T_1}^{T_2} \int_{\Omega} \sum_{j=0}^n \left(A_0 + \sum_{j=1}^n \omega_j A_j \right) (t, x) \chi_1(t, x) \chi_2(t, x) \times \\ \exp \left[i(\overline{R_2(t, x; \omega)} - R_1(t, x; \omega)) \right] dx dt + \dots \end{aligned}$$

which in turn leads to

$$\begin{aligned} \mathcal{I}_R = Ck \int_{T_1}^{T_2} \int_{\Omega} \left(A_0 + \sum_{j=1}^n \omega_j A_j \right) (t, x) \chi_1(t, x) \chi_2(t, x) \times \\ e^{-i \int_{-\infty}^{\frac{1}{2}(t+\omega \cdot x)} (A_0 + \sum_{j=1}^n \omega_j A_j)(t' + s, x' + s\omega) ds} dx dt + \dots \quad (28) \end{aligned}$$

where C is a constant and “ \dots ” represents terms of order $\mathcal{O}(1)$.

We turn now our attention to the left hand side of (18). Denoting by f and g the restrictions of u and v to $[T_1, T_2] \times \partial\Omega$, that is

$$f = u(t, x)|_{[-T_1, T_2] \times \partial\Omega} \quad g = v(t, x)|_{[-T_1, T_2] \times \partial\Omega},$$

we have by the Cauchy-Schwarz inequality

$$\begin{aligned} |\mathcal{I}_R| = \left| \langle (\Lambda_1 - \Lambda_2)(f), g \rangle_{[T_1, T_2] \times \partial\Omega} \right| \leq ||\Lambda_1 - \Lambda_2|| \times \\ ||f||_{H^1([T_1, T_2] \times \partial\Omega)} ||g||_{L^2([T_1, T_2] \times \partial\Omega)}. \end{aligned}$$

Using (21) the latter norm can be estimated by

$$\begin{aligned} ||g||_{L^2([T_1, T_2] \times \partial\Omega)} &= ||\chi_2(t, x)(1 + \mathcal{O}(k^{-1}))||_{L^2([T_1, T_2] \times \partial\Omega)} \\ &\leq ||\chi_2(t, x)||_{L^2([T_1, T_2] \times \partial\Omega)} + \mathcal{O}(k^{-1}), \end{aligned} \quad (29)$$

whereas by (25) the middle norm can be estimated by

$$\begin{aligned} ||f||_{H^1([T_1, T_2] \times \partial\Omega)} &\leq C \left[k ||\chi_1||_{L^2([T_1, T_2] \times \partial\Omega)} + \mathcal{O}(1) \right] \\ &= Ck \left[||\chi_1||_{L^2([T_1, T_2] \times \partial\Omega)} + \mathcal{O}(k^{-1}) \right]. \end{aligned} \quad (30)$$

In addition, since Ω is bounded and χ_j , $j = 1, 2$, is localized near a light ray, we have $\|\chi_j\|_{L^2([T_1, T_2] \times \partial\Omega)} \leq C$. Therefore, by (29) and (30)

$$|\mathcal{I}_R| \leq Ck \left[\|\Lambda_1 - \Lambda_2\| + \mathcal{O}(k^{-1}) \right]. \quad (31)$$

Dividing both sides of Green's formula (18) by k (i.e., (28) and (31)) and taking the limit as $k \rightarrow \infty$, we obtain via the triangle inequality and the change of coordinates $(t, x) = \sigma(1, \omega) + Y'$, $Y' \in \Pi_{(1, \omega)}$

$$\left| \int_{\Pi_{(1, \omega)}} \int_{\mathbb{R}} \left(A_0 + \sum_{j=1}^n \omega_j A_j \right) (Y' + \sigma(1, \omega)) \chi_1(Y') \chi_2(Y') \times \right. \\ \left. e^{-i \int_{-\infty}^{\sigma} (A_0 + \sum_{j=1}^n \omega_j A_j) (Y' + s(1, \omega)) ds} d\sigma dS_{Y'} \right| \leq C \|\Lambda_1 - \Lambda_2\|. \quad (32)$$

If we set

$$a(Y') := \int_{\mathbb{R}} \left(A_0 + \sum_{j=1}^n \omega_j A_j \right) (Y' + \sigma(1, \omega)) e^{-i \int_{-\infty}^{\sigma} (A_0 + \sum_{j=1}^n \omega_j A_j) (Y' + s(1, \omega)) ds} d\sigma,$$

equation (32) can be rewritten as

$$\left| \int_{\Pi_{(1, \omega)}} a(Y') \chi_1(Y') \chi_2(Y') dS_{Y'} \right| \leq C \|\Lambda_1 - \Lambda_2\|.$$

The conditions imposed on the support of χ_j , $j = 1, 2$, guarantee that the above estimate holds for any χ_j satisfying $\int_{\Pi_{(1, \omega)}} |\chi_j(Y')|^2 dS_{Y'} \leq 1$, thus a is a bounded linear functional on $L^1(\Pi_{(1, \omega)})$ and the estimate

$$\left| \int_{-\infty}^{\infty} \left(A_0 + \sum_{j=1}^n \omega_j A_j \right) (X' + \sigma(1, \omega)) \times \right. \\ \left. e^{i \int_{-\infty}^{\sigma} (A_0 + \sum_{j=1}^n \omega_j A_j) (X' + s(1, \omega)) ds} d\sigma \right| \leq C \|\Lambda_1 - \Lambda_2\|$$

holds. To finish the proof, we invoke the Fundamental Theorem of Calculus and rewrite the integral in the original coordinate system to obtain

$$\left| \exp \left[i \int_{-\infty}^{\infty} \left(A_0 + \sum_{j=1}^n \omega_j A_j \right) (t + s, x + s\omega) ds \right] - 1 \right| \leq C \|\Lambda_1 - \Lambda_2\|.$$

□

Corolary 3.2. *Let Λ_1, Λ_2 , represent the Dirichlet to Neumann operators for the hyperbolic equations (19), and let*

$$\alpha := \sup \left| \int_{-\infty}^{\infty} (A_0 + \sum_{j=0}^n \omega_j A_j)(t + s, x + s\omega) ds \right|$$

where the supremum is taken over $(t, x, \omega) \in [T_1, T_2] \times \Omega \times S^{n-1}$. If $\alpha < 2\pi$, then for all $(t, x) \in \mathbb{R}_{t,x}^{n+1}, \omega \in S^{n-1}$.

$$\left| \int_{-\infty}^{\infty} (A_0 + \sum_{j=1}^n \omega_j A_j)(t + s, x + s\omega) ds \right| \leq C ||| \Lambda_1 - \Lambda_2 |||, \quad (33)$$

where $||| \quad |||$ represents the operator norm between $H^1([T_1, T_2] \times \partial\Omega)$ and $L^2([T_1, T_2] \times \partial\Omega)$.

Proof. Denoting by β the integral $\int_{-\infty}^{\infty} (A_0 + \sum_{j=1}^n \omega_j A_j)(t + s, x + s\omega) ds$, we have

$$\frac{|e^{i\beta} - 1|}{|\beta|} = \frac{|\sin \frac{\beta}{2}|}{\frac{|\beta|}{2}}. \quad (34)$$

Since $\frac{|\beta|}{2} < \frac{\alpha}{2} < \pi$, the right hand side of (34) is bounded from below. It then follows that

$$\left| \int_{-\infty}^{\infty} (A_0 + \sum_{j=1}^n \omega_j A_j)(t + s, x + s\omega) ds \right| \leq C \left| e^{i \int_{-\infty}^{\infty} (A_0 + \sum_{j=1}^n \omega_j A_j)(t + s, x + s\omega) ds} - 1 \right|,$$

which in turn leads to (33). \square

To deal with the fact that uniqueness of the vector potentials is expected only up to a gauge transform we impose the divergence condition

$$\operatorname{div} \mathcal{A} = \partial_t A_0(t, x) + \sum_{j=1}^n \partial_{x_j} A_j(t, x) = 0. \quad (35)$$

By the remark after the definition of gauge equivalent pairs of potentials, we know that the difference of vector potentials is a the gradient of a scalar function. The divergence condition then implies that said scalar function must also be harmonic and hence equal to zero by the support conditions imposed on the vector potentials.

Denoting by F the ray transform of $A_0 + \sum_{j=1}^n \omega_j A_j$ along light rays $\gamma(t, x; \omega)$, we can rewrite (33) as

$$|F(t, x; \omega)| \leq C |||\Lambda_1 - \Lambda_2||| \quad (36)$$

for all $(t, x) \in \mathbb{R}_t \times \mathbb{R}_x^n$, $\omega \in S^{n-1}$. Taking the Fourier transform of F in the variables x_1, \dots, x_n yields

$$(\mathcal{F}_{(x \rightarrow \xi)} F(t, \cdot; \omega))(\xi) = \int_{\mathbb{R}^n} e^{-i\xi \cdot x} \int_{\mathbb{R}} \left(A_0 + \sum_{j=1}^n \omega_j A_j \right) (t + s, x + s\omega) \, ds \, dx,$$

and the change of coordinates $\tilde{x} = x + s\omega$, $\tilde{t} = t + s$, with Jacobian $|\frac{\partial(\tilde{t}, \tilde{x})}{\partial(t, x)}| = 1$ leads to

$$(\mathcal{F}_{(x \rightarrow \xi)} F(t, \cdot; \omega))(\xi) = e^{-i(\omega \cdot \xi)t} \int_{\mathbb{R}^n} \int_{\mathbb{R}} e^{-i\tilde{x} \cdot \xi} e^{-i(-\omega \cdot \xi)\tilde{t}} \left(A_0 + \sum_{j=1}^n \omega_j A_j \right) (\tilde{t}, \tilde{x}) \, d\tilde{t} \, d\tilde{x},$$

where the right hand side of the above equation is the Fourier transform (in all variables) of $A_0 + \sum_{j=1}^n \omega_j A_j$ at the point $(-\omega \cdot \xi, \xi)$. The above equation can be rewritten as

$$e^{it\omega \cdot \xi} (\mathcal{F}_{(x \rightarrow \xi)} F(t, \cdot; \omega))(\xi) = \left(A_0 + \sum_{j=1}^n \omega_j A_j \right)^\wedge (-\omega \cdot \xi, \xi)$$

and since the right hand side is independent of t , so is the left hand side. In particular when $t = 0$ we have

$$\left(A_0 + \sum_{j=1}^n \omega_j A_j \right)^\wedge (-\omega \cdot \xi, \xi) = (\mathcal{F}_{(x \rightarrow \xi)} F(0, \cdot; \omega))(\xi) =: G(\xi; \omega). \quad (37)$$

Since the potentials A_j are smooth and compactly supported, $F(0, \cdot; \cdot) : \mathbb{R}_x^n \times S^{n-1} \rightarrow \mathbb{R}$ is also smooth and compactly supported because for $|x|$ big enough, the light rays with direction $(1, \omega)$ emanating from the point $(0, x)$ do not intersect the support of the potentials A_j . Moreover by (36) it is uniformly bounded by $C |||\Lambda_1 - \Lambda_2|||$, and

$$\begin{aligned} |G(\xi; \omega)| &= \left| \int_{\mathbb{R}^n} e^{-ix \cdot \xi} F(0, x; \omega) \, dx \right| \\ &\leq \|F(0, \cdot; \cdot)\|_{L^\infty(\mathbb{R}_x^n \times S^{n-1})} \text{Vol}(B_n(R)) \\ &\leq CR^n |||\Lambda_1 - \Lambda_2||| \end{aligned} \quad (38)$$

shows that G is uniformly bounded in $\mathbb{R}_\xi^n \times S^{n-1}$.

Lemma 3.3. *Let Λ_1, Λ_2 , represent the Dirichlet to Neumann operators for the hyperbolic equations (19), and let α be as in corolary 3.2. If $\alpha < 2\pi$ and the divergence condition (35) holds, then*

$$|\widehat{A}_j(\tau, \xi)| \leq C |||\Lambda_1 - \Lambda_2|||, \quad 0 \leq j \leq n, \quad (39)$$

on the set $\{(\tau, \xi) : |\tau| \leq \frac{|\xi|}{2}\}$.

Proof. Proceeding as in the proof of theorem 3.3 in [23] (see also [24]), for (τ, ξ) fixed with $|\tau| < \frac{1}{2}|\xi|$ we can find unit vectors $\omega = \omega(\tau, \xi)$ parametrized by an $(n-2)$ -dimensional sphere with radius r , $\frac{\sqrt{3}}{2} \leq r \leq 1$, (we denote it by rS^{n-2}), satisfying $\tau + \omega(\tau, \xi) \cdot \xi = 0$, as well as $\omega(\theta\tau, \theta\xi) = \omega(\tau, \xi)$ for $\theta > 0$. In other words, we can find $\omega(\tau, \xi)$ homogenous of degree 0 in (τ, ξ) , such that $(\tau, \xi) \perp (1, \omega(\tau, \xi))$. If $n \geq 3$, we consider a maximal one dimensional sphere with radius r contained in rS^{n-2} and choose unit vectors $\omega^{(1)}(\tau, \xi), \dots, \omega^{(n)}(\tau, \xi)$ forming the vertices of a regular polygon with n sides. If $n = 2$ we let $\omega^{(1)}(\tau, \xi)$ and $\omega^{(2)}(\tau, \xi)$ be the only two elements of rS^0 . In both cases we then study the set of $n+1$ equations

$$\begin{cases} \widehat{A}_0(\tau, \xi) + \sum_{j=1}^n \omega_j^{(k)}(\tau, \xi) \widehat{A}_j(\tau, \xi) = G(\xi; \omega^{(k)}(\tau, \xi)), & k = 1, \dots, n \\ \frac{1}{\sqrt{\tau^2 + |\xi|^2}} \left(\tau \widehat{A}_0(\tau, \xi) + \sum_{j=1}^n \xi_j \widehat{A}_j(\tau, \xi) \right) = 0, \end{cases} \quad (40)$$

where the last equation is a simple consequence of the divergence condition (35). The left hand side of (40) can be expressed as $M(\tau, \xi) \hat{\mathcal{A}}(\tau, \xi)$, where

$$M(\tau, \xi) = \begin{pmatrix} 1 & \omega_1^{(1)}(\tau, \xi) & \dots & \omega_n^{(1)}(\tau, \xi) \\ 1 & \omega_1^{(2)}(\tau, \xi) & \dots & \omega_n^{(2)}(\tau, \xi) \\ \dots & \dots & \dots & \dots \\ 1 & \omega_1^{(n)}(\tau, \xi) & \dots & \omega_n^{(n)}(\tau, \xi) \\ \frac{\tau}{\sqrt{\tau^2 + |\xi|^2}} & \frac{\xi_1}{\sqrt{\tau^2 + |\xi|^2}} & \dots & \frac{\xi_n}{\sqrt{\tau^2 + |\xi|^2}} \end{pmatrix}$$

has homogeneous entries of degree 0 in (τ, ξ) . We claim that $M(\tau, \xi)$ is invertible. To prove this statement it suffices to show that the homogeneous

system

$$\left\{ \begin{array}{l} \widehat{A}_0(\tau, \xi) + \sum_{j=1}^n \omega_j^{(k)}(\tau, \xi) \widehat{A}_j(\tau, \xi) = 0, \quad k = 1, \dots, n \\ \frac{1}{\sqrt{\tau^2 + |\xi|^2}} \left(\tau \widehat{A}_0(\tau, \xi) + \sum_{j=1}^n \xi_j \widehat{A}_j(\tau, \xi) \right) = 0, \end{array} \right. \quad (41)$$

has no non-trivial solution. By theorem 3.4 in [23] (see also [24]), potentials satisfying the first n equations are those of the form

$$\begin{aligned} \widehat{A}_0(\tau, \xi) &= \tau \Phi(\tau, \xi), \\ \widehat{A}_j(\tau, \xi) &= \xi_j \Phi(\tau, \xi), \quad 1 \leq j \leq n, \end{aligned}$$

for some smooth function Φ . The last equation in (41) gives $\Phi(\tau, \xi) \sqrt{\tau^2 + |\xi|^2} = 0$, which in turn leads to $\Phi \equiv 0$, and $\widehat{\mathcal{A}} = 0$.

Since $M(\tau, \xi)$ is invertible we can write

$$\widehat{A}_j(\tau, \xi) = \sum_{k=1}^n c_{k,j}(\tau, \xi) G(\xi; \omega^{(k)}(\tau, \xi)), \quad 1 \leq k \leq n, \quad 0 \leq j \leq n,$$

for some $c_{k,j}(\tau, \xi)$ homogeneous of degree 0 in (τ, ξ) . It follows then that

$$\begin{aligned} |\widehat{A}_j(\tau, \xi)| &\leq \sum_{k=1}^n |c_{k,j}(\tau, \xi)| |G(\xi; \omega^{(k)}(\tau, \xi))| \\ &\leq C |||\Lambda_1 - \Lambda_2||| \sum_{k=1}^n |c_{k,j}(\tau, \xi)|, \end{aligned} \quad (42)$$

where in the last line of the previous inequality we used the uniform bound (38).

In view of the homogeneity of the functions $c_{k,j}(\tau, \xi)$ it suffices to work on the compact set $\{(\tau, \xi) : \tau^2 + |\xi|^2 = 1, |\tau| \leq \frac{|\xi|}{2}\}$. The entries of the inverse matrix of $M(\tau, \xi)$ have the form

$$c_{k,j}(\tau, \xi) = \frac{1}{\det M(\tau, \xi)} C_{j,k}(\tau, \xi)$$

where $C_{j,k}(\tau, \xi)$ is the (j, k) -cofactor of $M(\tau, \xi)$. Since the entries of $M(\tau, \xi)$ have absolute value less or equal to one, and since $C_{j,k}(\tau, \xi)$ consists of sums of products of n such entries, we have

$$|c_{k,j}(\tau, \xi)| \leq \frac{|C_{j,k}(\tau, \xi)|}{|\det M(\tau, \xi)|} \leq \frac{n}{|\det M(\tau, \xi)|}.$$

The quantity $|\det M(\tau, \xi)|$ represents the $(n+1)$ -dimensional volume generated by the vectors $\{(1, \omega^{(1)}(\tau, \xi)), \dots, (1, \omega^{(n)}(\tau, \xi)), (\tau, \xi)\}$. Due to our choice of $\omega^{(1)}(\tau, \xi), \dots, \omega^{(n)}(\tau, \xi)$ this volume does not depend on the point (τ, ξ) . Moreover, $|\det M(\tau, \xi)| = V \times P(\tau, \xi)$ where $P(\tau, \xi)$ is the projection of (τ, ξ) into the linear subspace generated by the set of vectors $\{(1, \omega^{(1)}(\tau, \xi)), \dots, (1, \omega^{(n)}(\tau, \xi))\}$ and V is the n -dimensional volume generated by these vectors. This projection is given by $C \sin \varphi$ where φ is the angle between (τ, ξ) and said subspace. Since the vectors $(1, \omega^{(k)}(\tau, \xi))$, $1 \leq k \leq n$, are located in the boundary of the light cone $\{(\tau, \xi) : |\tau| \geq |\xi|\}$, this angle is bounded below by $\frac{\pi}{8}$. Therefore the value $|\det M(\tau, \xi)|$ is uniformly bounded from below by $V \sin \frac{\pi}{8}$ on $\{(\tau, \xi) : \tau^2 + |\xi|^2 = 1, |\tau| \leq \frac{|\xi|}{2}\}$. Hence

$$|c_{k,j}(\tau, \xi)| \leq \frac{n}{V \sin \frac{\pi}{8}},$$

and by (42) we obtain the uniform estimate

$$|\widehat{A_j}(\tau, \xi)| \leq C |||\Lambda_1 - \Lambda_2|||$$

on the set $\{(\tau, \xi) : |\tau| \leq \frac{|\xi|}{2}\}$. \square

The following statement is a result about harmonic measures, its proof can be found in [2].

Lemma 3.4. *Consider the strip*

$$S = \{z = z_1 + iz_2 : z_1 \in \mathbb{R}, |z_2| < 2|\tau_0|\pi, \tau_0 \neq 0\}$$

and the rays

$$p_1 = \{z : -\infty < z_1 \leq -2|\tau_0|, z_2 = 0\}, \quad p_2 = \{z : 2|\tau_0| \leq z_1 < \infty, z_2 = 0\}$$

in the complex plane \mathbb{C} .

If $E = p_1 \cup p_2$ and $G = S \setminus E$ is the strip with cuts along the rays p_1 and p_2 , we have

$$\frac{2}{3} < \varpi(z, E, G) \leq 1, \tag{43}$$

where $\varpi(z, G, E)$ is the harmonic measure of E with respect to G . More precisely

$$\varpi(\zeta) = \frac{1}{\pi} \int_{-\infty}^{\infty} \chi_{E'}(t) \frac{\zeta_2}{(t - \zeta_1)^2 + \zeta_2^2} dt, \quad (44)$$

where $\chi_{E'}(t)$ is the characteristic function of the set $E' = \{t \in \mathbb{R} : |t| \leq 1\} \cup \{t \in \mathbb{R} : |t| > e\}$.

We now perform a rotation in ξ space to make any given vector $(\tau, \xi) = (\tau, \xi_1, \dots, \xi_{n-1}, \xi_n)$ have the representation $(\tau, 0, \dots, 0, \nu)$. Based on the previous statements we want to ‘embed’ the ν -axis into a strip in the complex plane and use the bounds developed in the previous lemma.

Lemma 3.5. *Let Λ_1, Λ_2 , represent the Dirichlet to Neumann operators for the hyperbolic equations (19), and let α be as in corolary 3.2. If $\alpha < 2\pi$ and the divergence condition (35) holds, then on the set $\{(\tau, \xi) : |\tau| > \frac{|\xi|}{2}\}$ we have*

$$|\widehat{A_j}(\tau, \xi)| \leq C \frac{e^{\frac{2|\tau|a}{3}} ||\Lambda_1 - \Lambda_2||^{\frac{2}{3}}}{|\tau|^{\frac{1}{3}}}, \quad (45)$$

where a is some positive number bigger than the diameter of Ω .

Proof. Since the potentials A_j , $0 \leq j \leq n$, are compactly supported, the functions $\widehat{A_j}(\tau_0, 0, \dots, 0, \nu)$ admit an analytic extension in ν into the complex plane. Letting

$$\begin{aligned} \Pi &= \{\nu = (\nu_1, \nu_2) : \nu_1 \in \mathbb{R}, |\nu_2| < 2|\tau_0|\pi, \tau_0 \neq 0\}, \\ q_1 &= \{\nu = (\nu_1, \nu_2) : -\infty < \nu_1 \leq -2|\tau_0|, \nu_2 = 0\}, \\ q_2 &= \{\nu = (\nu_1, \nu_2) : 2|\tau_0| \leq \nu_1 < \infty, \nu_2 = 0\} \end{aligned}$$

and restricting the potentials to the ν -axis, (43) leads to

$$\frac{2}{3} < \varpi(\nu, E_1, G_1) \leq 1,$$

where $E_1 = q_1 \cup q_2$ and $G_1 = \Pi \setminus E_1$. Denoting by $v_j(\nu) = \widehat{A_j}(2\tau_0, 0, \dots, 0, \nu)$, the above restriction we have by the two-constant theorem (see [18] Theorem 9.4.5)

$$|v_j(\nu)| \leq m_j^{\frac{2}{3}} M_j^{\frac{1}{3}} \quad (46)$$

where m_j and M_j are the respective upper bounds of the modulus of $v(\nu)$ on the rays q_1 and q_2 and on the union of the lines $q'_1 = \{(\nu_1, \nu_2) : \nu_1 \in \mathbb{R}, \nu_2 = -2|\tau_0|\pi\}$ and $q'_2 = \{(\nu_1, \nu_2) : \nu_1 \in \mathbb{R}, \nu_2 = 2|\tau_0|\pi\}$. We point out that the rays q_1 and q_2 are contained in the set $\{(\tau, \xi) : |\tau| \leq \frac{|\xi|}{2}\}$ and that (39) provides an estimate for $|v_j(\nu)|$ in that region. To compute M_j we resort to the equality

$$v_j(\nu) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i(\nu_1 + i\nu_2)x_n} W_j(2\tau_0, 0, \dots, 0, x_n) dx_n$$

where W_j is the Fourier transform of A_j in all variables except x_n . These functions are compactly supported in x_n and the above integrand is nonzero only on a bounded subset of the real numbers. Hence on $q'_1 \cup q'_2$

$$|v_j(\nu)| \leq \frac{1}{2\pi} \sup_{x_n \in (-a(\Omega), a(\Omega))} |W_j(2\tau_0, 0, \dots, 0, x_n)| \int_{-\tilde{a}(\Omega)}^{\tilde{a}(\Omega)} e^{2|\tau_0|\pi x_n} dx_n,$$

where \tilde{a} is a positive number bigger than $\text{diam}(\Omega)$. Integration in x_n then leads to

$$|v_j(\nu)| \leq C \frac{e^{2|\tau_0|a}}{|\tau_0|}$$

where $\nu \in q'_1 \cup q'_2$ and $a = \tilde{a}\pi$. Therefore, when ν is a real number satisfying $-2|\tau_0| < \nu < 2|\tau_0|$ we have by (46)

$$|v_j(\nu)| \leq C \frac{e^{\frac{2|\tau_0|a}{3}} m_j^{\frac{2}{3}}}{|\tau_0|^{\frac{1}{3}}}.$$

The above arguments work for any line contained in the hyperplane $\tau = \tau_0$ that passes through the origin. Hence by (39), for $\{|\tau| > \frac{|\xi|}{2}\}$ we have

$$|\widehat{A_j}(\tau, \xi)| \leq C \frac{e^{\frac{2|\tau|a}{3}} \|\Lambda_1 - \Lambda_2\|^{\frac{2}{3}}}{|\tau|^{\frac{1}{3}}}.$$

□

We can now establish the desired stability estimate for the vector potentials. The general idea is to use the inequality $\|f\|_{L^\infty} \leq C \|\widehat{f}\|_{L^1}$ and partition $\mathbb{R}_\tau \times \mathbb{R}_\xi^n$ in an appropriate way.

Theorem 3.6. *Suppose that the vector and scalar potentials $\mathcal{A}^{(l)} = (A_0^{(l)}, \dots, A_n^{(l)})$, $V^{(l)}$, $l = 1, 2$, are real valued, compactly supported and C^∞ in t and x . Let $\mathcal{A} = (A_0, A_1, \dots, A_n)$ where $A_j = A_j^{(1)} - A_j^{(2)}$ and suppose that the following divergence condition holds*

$$\operatorname{div} \mathcal{A} = \partial_t A_0(t, x) + \sum_{j=1}^n \partial_{x_j} A_j(t, x) = 0,$$

and that the entries of the vector potential satisfy

$$\sup \left| \int_{-\infty}^{\infty} (A_0 + \sum_{j=0}^n \omega_j A_j)(t + s, x + s\omega) ds \right| < 2\pi,$$

where the supremum is taken over $(t, x; \omega) \in [T_1, T_2] \times \Omega \times S^{n-1}$.

If Λ_l represents the Dirichlet to Neumann operator associated to the hyperbolic problem (1)-(4), then the stability estimate

$$\max_{0 \leq j \leq n} \left\| A_j^{(1)}(t, x) - A_j^{(2)}(t, x) \right\|_{L^\infty(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C \left[\log \frac{1}{|||\Lambda_1 - \Lambda_2|||} \right]^{-1} \quad (47)$$

holds for Λ_1, Λ_2 satisfying $|||\Lambda_1 - \Lambda_2||| \ll 1$.

Proof. Let α be as in corollary 3.2. Since $\alpha < 2\pi$, from the Fourier inversion formula we have

$$A_j(t, x) = \frac{1}{(2\pi)^{n+1}} \iint_{\mathbb{R}_\tau \times \mathbb{R}_\xi^n} e^{i(t\tau + x \cdot \xi)} \widehat{A_j}(\tau, \xi) d\tau d\xi. \quad (48)$$

Taking absolute values we have for $\rho > 0$

$$\begin{aligned} |A_j(t, x)| &\leq \frac{1}{(2\pi)^{n+1}} \iint_{\mathbb{R}_\tau \times \mathbb{R}_\xi^n} |\widehat{A_j}(\tau, \xi)| d\tau d\xi \\ &\leq \frac{1}{(2\pi)^{n+1}} \iint_{B(\rho_1)} |\widehat{A_j}(\tau, \xi)| d\tau d\xi \\ &\quad + \frac{1}{(2\pi)^{n+1}} \iint_{B(\rho_1)^c} |\widehat{A_j}(\tau, \xi)| d\tau d\xi \\ &= I_1 + I_2, \end{aligned}$$

where $B(\rho)$ denotes the $(n+1)$ -dimensional ball $B(\rho) = \{(\tau, \xi) : |\tau|^2 + |\xi|^2 \leq \rho^2\}$. Since for $0 \leq j \leq n$, the potentials A_j , are C_0^∞ in t and x , for any $\beta > 0$, $\rho_1 > 0$, if $|\tau|^2 + |\xi|^2 \geq \rho_1^2$ we have

$$|\widehat{A_j}(\tau, \xi)| \leq \frac{C}{(|\tau|^2 + |\xi|^2)^{\frac{\beta}{2}}},$$

where C depends on the derivatives of $A_j(t, x)$ up to order β . When $\beta > n+1$, the integral I_2 converges. Moreover, when $\beta > n+2$ and $\rho > 1$, the following estimate holds

$$I_2 = \iint_{B(\rho)^c} |\widehat{A_j}(\tau, \xi)| d\tau d\xi \leq \frac{C}{\rho^{\beta-n-1}} \leq \frac{C}{\rho}. \quad (49)$$

To estimate I_1 we break up the ball $B(\rho)$ into two smaller pieces

$$\mathcal{C}_1 = B(\rho) \cap \left\{(\tau, \xi) : |\tau| < \frac{|\xi|}{2}\right\} \quad \text{and} \quad \mathcal{C}_2 = B(\rho) \cap \left\{(\tau, \xi) : |\tau| \geq \frac{|\xi|}{2}\right\}.$$

Then

$$I_1 \leq \iint_{\mathcal{C}_1} |\widehat{A_j}(\tau, \xi)| d\tau d\xi + \iint_{\mathcal{C}_2} |\widehat{A_j}(\tau, \xi)| d\tau d\xi,$$

and since \mathcal{C}_1 is a subset of $B(\rho)$ we have

$$I_1 \leq C\rho^{n+1}|||\Lambda_1 - \Lambda_2||| + \iint_{\mathcal{C}_2} |\widehat{A_j}(\tau, \xi)| d\tau d\xi.$$

With this decomposition, \mathcal{C}_2 is contained in the set $\{(\tau, \xi) : |\tau| > \frac{|\xi|}{2}\}$. Thus by (45)

$$I_2 \leq C\rho^{n+1}|||\Lambda_1 - \Lambda_2||| + C' e^{\frac{2\rho a}{3}} |||\Lambda_1 - \Lambda_2|||^{\frac{2}{3}} \rho^{n+\frac{2}{3}}. \quad (50)$$

Equations (48)-(50) lead to

$$|A_j(t, x)| \leq C \left[\frac{1}{\rho} + \rho^{n+1}|||\Lambda_1 - \Lambda_2||| + \rho^{n+\frac{2}{3}} e^{\frac{2\rho a}{3}} |||\Lambda_1 - \Lambda_2|||^{\frac{2}{3}} \right] \quad (51)$$

The rest of the proof is fairly standard. First we seek to impose a condition on $|||\Lambda_1 - \Lambda_2|||$ so that the the third term in the right hand side of (51) dominates the second one. This can be done by simple minimization in ρ of the function

$\frac{e^{2\rho a}}{\rho}$ over the interval $[1, +\infty)$. If $a < \frac{1}{2}$ we want $|||\Lambda_1 - \Lambda_2||| < 2ae$ and if $a \geq \frac{1}{2}$ then $|||\Lambda_1 - \Lambda_2||| < e^{2a}$. In both cases, if $|||\Lambda_1 - \Lambda_2||| < 1$ then

$$|A_j(t, x)| \leq C \left[\frac{1}{\rho} + \rho^{n+\frac{2}{3}} e^{\frac{2\rho a}{3}} |||\Lambda_1 - \Lambda_2|||^{\frac{2}{3}} \right]. \quad (52)$$

The next step is to choose ρ so that the two terms in the the right hand side of (52) are comparable. In other words we want ρ to satisfy the identity

$$\frac{C}{\rho} = \rho^{n+\frac{2}{3}} e^{\frac{2\rho a}{3}} |||\Lambda_1 - \Lambda_2|||^{\frac{2}{3}}$$

for some constant C . Taking logarithms on both sides of the previous equation yields the following equivalent identity

$$2 \log \frac{C}{|||\Lambda_1 - \Lambda_2|||} = (3n + 5) \log \rho + 2a\rho, \quad (53)$$

where the right hand side of (53) is one to one when $\rho > 0$ and hence it admits a unique solution. On the other hand, the inequality $\log \rho \leq \rho$ for positive ρ as well as (53) lead to

$$2 \log \frac{C}{|||\Lambda_1 - \Lambda_2|||} \leq (3n + 5 + 2a)\rho,$$

or

$$\frac{1}{\rho} \leq \frac{3n + 5 + 2a}{2} \left[\log \frac{C}{|||\Lambda_1 - \Lambda_2|||} \right]^{-1},$$

and (52) becomes

$$|A_j(t, x)| \leq C'' \left[\log \frac{C'}{|||\Lambda_1 - \Lambda_2|||} \right]^{-1} \leq C \left[\log \frac{1}{|||\Lambda_1 - \Lambda_2|||} \right]^{-1},$$

where C depends on n , Ω and derivatives of $A_j(t, x)$ for $0 \leq j \leq n$. \square

4 Stability of the scalar potentials

In this section we establish a log-log type estimate for the scalar potentials. We point out that the estimate from theorem 3.6 is independent of the scalar

potentials. This is because the term involving the difference of said potentials is not the leading term in the asymptotics (28) and it does not survive the process of dividing by k and taking the limit as $k \rightarrow +\infty$. In the following lines, we reuse the techniques developed in the previous sections while following closely the ideas from Isakov and Sun in [15].

Theorem 4.1. *Suppose that the vector and scalar potentials $\mathcal{A}^{(l)} = (A_0^{(l)}, \dots, A_n^{(l)})$, $V^{(l)}$, $l = 1, 2$, are real valued, compactly supported and C^∞ in t and x . Let $V = V^{(1)} - V^{(2)}$, $\mathcal{A} = (A_0, A_1, \dots, A_n)$, where $A_j = A_j^{(1)} - A_j^{(2)}$ and suppose that the following divergence condition holds*

$$\operatorname{div} \mathcal{A} = \partial_t A_0(t, x) + \sum_{j=1}^n \partial_{x_j} A_j(t, x) = 0,$$

and that the entries of the vector potential satisfy

$$\sup \left| \int_{-\infty}^{\infty} (A_0 + \sum_{j=1}^n \omega_j A_j)(t + s, x + s\omega) ds \right| < 2\pi,$$

where the supremum is taken over $(t, x; \omega) \in [T_1, T_2] \times \Omega \times S^{n-1}$.

If Λ_l represents the Dirichlet to Neumann operator associated to the hyperbolic problem (1)-(4), then for Λ_1, Λ_2 satisfying $|||\Lambda_1 - \Lambda_2||| \ll 1$, the following stability estimates hold

$$\begin{aligned} |||\mathcal{A}|||_0 &\leq C \left(\log \frac{1}{|||\Lambda_1 - \Lambda_2|||} \right)^{-1}, \\ ||V||_{L^\infty(\mathbb{R}_t \times \mathbb{R}_x^n)} &\leq C \left(\log \left(\log \frac{1}{|||\Lambda_1 - \Lambda_2|||} \right) \right)^{-1}, \end{aligned}$$

where

$$|||\mathcal{A}|||_0 = |||\mathcal{A}^{(1)} - \mathcal{A}^{(2)}|||_0 := \max_{0 \leq j \leq n} \|A_j^{(1)}(t, x) - A_j^{(2)}(t, x)\|_{L^\infty(\mathbb{R}_t \times \mathbb{R}_x^n)}.$$

Proof. In view of our previous results, it is enough to obtain a uniform estimate for the X-ray transform along light rays of the difference of the scalar potentials. By theorem 3.6, for arbitrary smooth compactly supported scalar potentials $V^{(1)} \neq V^{(2)}$, we have

$$|||\mathcal{A}|||_0 \leq C \left(\log \frac{1}{|||\Lambda_1 - \Lambda_2|||} \right)^{-1}. \quad (54)$$

Green's formula (18) with u and v solutions of the forward and backward hyperbolic problem respectively, as well as the triangle inequality give

$$\begin{aligned}
|\langle (V^{(1)} - V^{(2)})u, v \rangle_{[T_1, T_2] \times \Omega}| &\leq |\langle (\Lambda_1 - \Lambda_2)u, v \rangle_{[T_1, T_2] \times \partial\Omega}| \\
&+ \sum_{j=0}^n |\langle (A_j^{(1)} - A_j^{(2)})u, (-i\partial_{x_j}v) \rangle_{[T_1, T_2] \times \Omega}| \\
&+ \sum_{j=0}^n |\langle (A_j^{(1)} - A_j^{(2)})(-i\partial_{x_j}u), v \rangle_{[T_1, T_2] \times \Omega}| \\
&+ \sum_{j=0}^n |\langle [(A_j^{(2)})^2 - (A_j^{(1)})^2]u, v \rangle_{[T_1, T_2] \times \Omega}|. \quad (55)
\end{aligned}$$

When u and v are given by the GO anzats developed in section 2, the discussion of the assymptotics of the derivatives $\partial_{x_j}u, \partial_{x_j}v$, $0 \leq j \leq n$, give the estimates

$$\begin{aligned}
|\langle (\Lambda_1 - \Lambda_2)u, v \rangle_{[T_1, T_2] \times \partial\Omega}| &\leq Ck(\|\Lambda_1 - \Lambda_2\| + \mathcal{O}(k^{-1})), \\
|\langle (A_j^{(1)} - A_j^{(2)})u, (-i\partial_{x_j}v) \rangle_{[T_1, T_2] \times \Omega}| &\leq Ck(\|\mathcal{A}\|_0 + \mathcal{O}(k^{-1})), \\
|\langle (A_j^{(1)} - A_j^{(2)})(-i\partial_{x_j}u), v \rangle_{[T_1, T_2] \times \Omega}| &\leq Ck(\|\mathcal{A}\|_0 + \mathcal{O}(k^{-1})), \\
|\langle [(A_j^{(2)})^2 - (A_j^{(1)})^2]u, v \rangle_{[T_1, T_2] \times \Omega}| &\leq C\|\mathcal{A}\|_0,
\end{aligned}$$

where the last inequality follows from the fact that $|A_j^{(1)}(t, x) - A_j^{(2)}(t, x)| \leq C$ for all $(t, x) \in \mathbb{R}_t \times \mathbb{R}_x^n$.

On the other hand, the LHS of (55) gives

$$\begin{aligned}
\left| \int_{T_1}^{T_2} \int_{\Omega} V(t, x) u(t, x) \overline{v(t, x)} dx dt \right| &= \left| \int_{T_1}^{T_2} \int_{\Omega} V(t, x) \chi_1(t, x) \chi_2(t, x) \times \right. \\
&\quad \left. e^{-i \int_{-\infty}^{\frac{1}{2}(t+\omega \cdot x)} (A_0 + \sum_{j=1}^n \omega_j A_j)(t'+s, x'+s\omega) ds} dx dt + \dots \right| \quad (56)
\end{aligned}$$

where (t', x') is the projection of (t, x) onto $\Pi_{(1, \omega)}$ and “ \dots ” represents terms of order $\mathcal{O}(k^{-1})$. Also, a simple analysis of e^{iz} for small $|z|$ gives

$$e^{-i \int_{-\infty}^{\frac{1}{2}(t+\omega \cdot x)} (A_0 + \sum_{j=1}^n \omega_j A_j)(t'+s, x'+s\omega) ds} = 1 + \mathcal{O}(\|\mathcal{A}\|_0)$$

and thus (55) leads to

$$\left| \int_{T_1}^{T_2} \int_{\Omega} V(t, x) \chi_1(t, x) \chi_2(t, x) dx dt \right| \leq C_1 k (||| \mathcal{A} |||_0 + ||| \Lambda_1 - \Lambda_2 |||) + C_2 ||| \mathcal{A} |||_0 + \mathcal{O}(k^{-1})$$

As in previous cases, the fact that the functions χ_j are supported near light rays shows that for $k > 0$ the following estimate holds

$$\left| \int_{-\infty}^{\infty} V(t + s, x + s\omega) ds \right| \leq C_1 k (||| \mathcal{A} |||_0 + ||| \Lambda_1 - \Lambda_2 |||) + C_2 ||| \mathcal{A} |||_0 + \frac{C_3}{k}. \quad (57)$$

Next we choose k so that the first and last terms in the previous equation are comparable in size. To this end, let

$$k = (||| \Lambda_1 - \Lambda_2 ||| + ||| \mathcal{A} |||_0)^{-\frac{1}{2}},$$

then

$$(||| \Lambda_1 - \Lambda_2 ||| + ||| \mathcal{A} |||_0) k = \frac{1}{k} \leq C (||| \Lambda_1 - \Lambda_2 ||| + ||| \mathcal{A} |||_0)^{\frac{1}{2}},$$

and (57) gives

$$\begin{aligned} \left| \int_{-\infty}^{\infty} V(t + s, x + s\omega) ds \right| &\leq C_1 (||| \Lambda_1 - \Lambda_2 ||| + ||| \mathcal{A} |||_0)^{\frac{1}{2}} + C_2 ||| \mathcal{A} |||_0 \\ &\leq C (||| \mathcal{A} |||_0^{\frac{1}{2}} + ||| \Lambda_1 - \Lambda_2 |||^{\frac{1}{2}}), \end{aligned}$$

where the last inequality holds when both $||| \Lambda_1 - \Lambda_2 |||, ||| \mathcal{A} |||_0 < 1$ (recall that if $0 < \epsilon < 1$, then $\epsilon < \sqrt{\epsilon} < 1$). Estimate (54) then gives

$$\begin{aligned} \left| \int_{-\infty}^{\infty} V(t + s, x + s\omega) ds \right| &\leq C \left[\left(\log \frac{1}{||| \Lambda_1 - \Lambda_2 |||} \right)^{-\frac{1}{2}} + ||| \Lambda_1 - \Lambda_2 |||^{\frac{1}{2}} \right] \\ &\leq C \left(\log \frac{1}{||| \Lambda_1 - \Lambda_2 |||} \right)^{-\frac{1}{2}}. \end{aligned} \quad (58)$$

As in section 3 we get from (58)

$$\begin{aligned} ||| V |||_{L^\infty(\mathbb{R}_t \times \mathbb{R}_x^n)} &\leq C' \left(\log \frac{1}{C \left(\log \frac{1}{||| \Lambda_1 - \Lambda_2 |||} \right)^{-1/2}} \right)^{-1} \\ &\leq C \left(\log \left(\log \frac{1}{||| \Lambda_1 - \Lambda_2 |||} \right) \right)^{-1}. \end{aligned}$$

□

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